

AUTOMORPHIC FORMS AND DIFFERENTIABILITY PROPERTIES

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ABSTRACT. We consider Fourier series given by a type of fractional integral of automorphic forms, and we study their local and global properties, especially differentiability and fractal dimension of the graph of their real and imaginary parts. In this way we can construct fractal objects and continuous non-differentiable functions associated with elliptic curves and theta functions.

§1. INTRODUCTION

“Riemann’s example”, the real function defined as

$$F(x) = \sum_{n=1}^{\infty} \frac{\sin(2\pi n^2 x)}{n^2},$$

is a trigonometric series whose history that can be traced back to the origins of the theory of functions in the 19th century. Apparently F was initially introduced as an example of a continuous but nowhere differentiable function, and it has motivated a vast literature (see the references in [2]) culminating in several works proving that F is a fractal-like continuous function which is differentiable only at rational values of the form $a/(2b)$ with a and b odd integers. According to Weierstrass [16], this latter fact contradicted a claim of Riemann (but the extensive historical analysis in [2] does not support the claim of Weierstrass).

In this paper we shall present general results that allow us to determine the differentiability and other local and global properties of a wide class of functions. In this setting Riemann’s example will be only a simple particular case (see the last comments in §2).

In order to illustrate the diversity of the functions considered, we mention the following examples:

$$a) \quad F_1(x) = \sum_{n \equiv \pm 1 \pmod{12}} \frac{\sin(2\pi n^2 x)}{n^2} - \sum_{n \equiv \pm 5 \pmod{12}} \frac{\sin(2\pi n^2 x)}{n^2},$$

where $n \equiv a \pmod{b}$ means “ b divides $n - a$ ”.

$$b) \quad F_2(x) = \sum_{n=1}^{\infty} \frac{a_n}{n^{8/5}} \sin(2\pi n x),$$

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where a_n are the coefficients of the Hasse-Weil L -function of the elliptic curve $E : y^2 = x^3 + 1$.

$$\text{c) } F_3(x) = \sum_{n=1}^{\infty} \frac{c_n}{n^{20/3}} \cos(2\pi nx),$$

where c_n is the n -th coefficient in the power series expansion of $x \prod_{k=1}^{\infty} (1 - x^k)^{24}$.

$$\text{d) } F_4(x) = \sum_{n=1}^{\infty} \frac{r(n)}{n^{5/4}} e^{2\pi i n x},$$

where $r(n) = |\{(x, y) \in \mathbf{Z}^2 : n = x^2 + 2y^2\}|$.

A direct consequence of our main results is that F_1, F_2 and F_3 are continuous functions which are differentiable in \mathbf{Q} and non-differentiable on $\mathbf{R} - \mathbf{Q}$; moreover the graphs of these functions are fractals with dimensions $5/4, 7/5$ and $4/3$, respectively.

Finer properties can be specified in each case, for instance:

a) F_1 is not only a simple example of a continuous function having \mathbf{Q} as its set of differentiability (such examples usually do not appear in calculus textbooks) but also its derivative vanishes on \mathbf{Q} .

b) The value of the derivative of F_2 on \mathbf{Q} is related to special values of the Hasse-Weil L -function of E and its twists. To illustrate this fact in an arithmetic way, we mention that if $8/5$ is replaced by 2 in the definition of F_2 , and E by an arbitrary elliptic curve over \mathbf{Q} , then E contains finitely many rational points whenever the resulting function has a non-vanishing derivative at $x = 0$.

c) The remarkable fact about F_3 is that the size of the coefficients c_n remained as a conjecture from the time of Ramanujan to the celebrated work of P. Deligne on the Riemann hypothesis over finite fields. Nevertheless with comparatively simple arguments, some analytic properties of F_3 can be proved. For instance, $F_3 \in \Lambda_{\beta}$ for $\beta < 2/3$ and $F_3(a/b + h) - F_3(a/b) = ch + O(|h|^{4/3})$, for $a/b \in \mathbf{Q}$.

d) Finally, F_4 has a more complicated differentiability set. Namely, it is not differentiable on $\mathbf{R} - \mathbf{Q}$ and it is differentiable at $x_0 = a/b$ (a reduced fraction) if and only if the remainder when b is divided by 8 is even and non-zero.

The common aspect in these apparently unrelated examples is that each is a kind of fractional integral of the Fourier expansion (at $i\infty$) of some automorphic form. Namely, F_1 corresponds to the η -function, F_2 to the newform associated to the modular curve $E : y^2 = x^3 + 1$, F_3 to the discriminant function (c_n is the Ramanujan τ -function) and F_4 to a 2-dimensional θ -function. In this paper we shall use number theoretical tools (especially the theory of automorphic forms) to determine, in general, differentiability, Lipschitz and fractal properties in this setting.

The contents of the subsequent sections are as follows: In §2 we introduce some basic notation and we state our main results, which are proved in §4 (after introducing some auxiliary results in §3). We devote §5 to a closer study of the Fourier series arising from theta functions. Finally, §6 can be considered as an appendix including some computer graphics with comments.

§2. NOTATION AND MAIN RESULTS

We shall denote by $\mathcal{M}_r(\Gamma, m)$ the space of automorphic forms of weight $r > 0$ (not necessarily integral) with respect to the Fuchsian group Γ and having multiplier

system m (see [10]), *i.e.*, if $f \in \mathcal{M}_r(\Gamma, m)$, we have

$$f(\gamma z) = m(\gamma)(j_\gamma(z))^r f(z) \quad \text{for every } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

At each cusp \mathfrak{a} of Γ , f admits an expansion (see [10])

$$f(\sigma_{\mathfrak{a}} z) = e(\kappa_{\mathfrak{a}} z)(j_{\sigma_{\mathfrak{a}}}(z))^r \sum_{n=0}^{\infty} a_n^{\mathfrak{a}} e(nz),$$

where $\sigma_{\mathfrak{a}}$ is the scaling matrix and $e(t)$ is an abbreviation for $e^{2\pi i t}$. The constants $0 \leq \kappa_{\mathfrak{a}} < 1$ and $a_n^{\mathfrak{a}}$ depend only on the equivalence class of \mathfrak{a} under Γ .

We shall say that $f \in \mathcal{M}_r(\Gamma, m)$ is cuspidal at $x \in \mathbf{R} \cup \{i\infty\}$ if $\mathfrak{a} = x$ is a cusp for Γ and $a_0^{\mathfrak{a}} = 0$ whenever $\kappa_{\mathfrak{a}} = 0$. If f is cuspidal at every cusp, then f is said to be a cusp form, and we shall write $f \in \mathcal{S}_r(\Gamma, m)$. Usually, we shall abbreviate $\mathcal{M}_r(\Gamma, m)$ and $\mathcal{S}_r(\Gamma, m)$ by \mathcal{M}_r and \mathcal{S}_r .

In this paper we shall consider Γ to be a subgroup of $SL_2(\mathbf{Z})$ of finite index containing the unit translation $\gamma(z) = z + 1$ with $m(\gamma) = 1$. Under these conditions the cusps for Γ are $\mathbf{Q} \cup \{i\infty\}$, and any $f \in \mathcal{M}_r$ admits a Fourier expansion

$$f(z) = \sum_{n=0}^{\infty} a_n e(nz).$$

For $f \in \mathcal{M}_r - \{0\}$ and $x \in \mathbf{R}$ we consider the following scaled α -fractional integral of f , when it converges:

$$f_{\alpha}(x) = \sum_{n=1}^{\infty} \frac{a_n}{n^{\alpha}} e(nx).$$

It turns out that f_{α} is a well defined continuous function if and only if $f \in \mathcal{S}_r$ with $\alpha > r/2$ or $f \in \mathcal{M}_r - \mathcal{S}_r$ with $\alpha > r$ (see Proposition 3.1). Consequently we shall refer to these conditions as convergence conditions.

Throughout this paper, the letters C and K with different subscripts and superscripts will be used to denote constants (positive if the contrary is not explicitly indicated, and not necessarily the same on each appearance).

The symbols $W^{n,p} = W^{n,p}([0, 1])$, $\Lambda_{\beta} = \Lambda_{\beta}([0, 1])$, $\lambda_{\beta} = \lambda_{\beta}([0, 1])$ will denote the corresponding Sobolev space (see [1]) and Lipschitz spaces (see II.3 in [17]). The localization of these latter in x_0 will be represented by $\Lambda_{\beta}(x_0)$ and $\lambda_{\beta}(x_0)$, *i.e.*, in Landau notation, which we shall use extensively,

$$\begin{aligned} g \in \Lambda_{\beta}(x_0) &\Leftrightarrow g(x) - g(x_0) = O(|x - x_0|^{\beta}), \\ g \in \lambda_{\beta}(x_0) &\Leftrightarrow g(x) - g(x_0) = o(|x - x_0|^{\beta}) \end{aligned}$$

as $x \rightarrow x_0$. In the same way, Λ_{β}^{\log} will indicate the space of functions such that $g(x) - g(y) = O(|x - y|^{\beta} |\log |x - y||)$ as $x \rightarrow y$, uniformly on y , and $\Lambda_{\beta}^{\log}(x_0)$ will represent its localization at x_0 .

Finally we shall introduce some concepts related to fractal dimension.

Given a continuous function $g : [0, 1] \rightarrow \mathbf{R}$, we define its h -variation, $0 < h < 1$, in an interval $I \subset [0, 1]$ as

$$V(g, h; I) = \sup_{x_j, y_j} \sum_j |g(x_j) - g(y_j)|,$$

where $x_j, y_j \in [(j-1)h, jh] \cap I$, $j = 1, 2, 3, \dots$

We shall abbreviate $V(g, h; [0, 1])$ by $V(g, h)$. Note that the number of cells of the lattice $h\mathbf{Z} \times h\mathbf{Z}$ needed to cover the graph, \mathcal{G} , of g differs from $h^{-1}V(g, h)$ by $O(h^{-1})$, and hence the fractal (Minkowski, box counting) dimension of \mathcal{G} is given by

$$\dim(\mathcal{G}) = \lim_{h \rightarrow 0^+} \frac{\log(h^{-1} + h^{-1}V(g, h))}{-\log h}.$$

Even if this limit does not exist, we can always define upper and lower dimensions, $\overline{\dim}(\mathcal{G})$ and $\underline{\dim}(\mathcal{G})$, by changing \lim to $\overline{\lim}$ and $\underline{\lim}$.

Now we state our main results. As before, f will represent an element of $\mathcal{M}_r - \{0\}$ and it will be implicitly assumed that convergence conditions are fulfilled.

Theorem 2.1. *If $f_\alpha \in \lambda_\beta(x_0)$ with $\beta = \alpha - r/2 < 1$, then f is cuspidal at x_0 . Moreover, if f is a cusp form, $f_\alpha \in \Lambda_\beta^{\log} - \lambda_\beta$.*

Corollary 2.1.1. *For $\alpha < r/2 + 1$, f_α is not differentiable at irrationals.*

Remark. This result cannot be extended to $\alpha > r/2 + 1$ because Lemma 3.2 c), d) implies that, in this range, $f_\alpha \in W^{1,2}$ for $f \in \mathcal{M}_r$ with $r < 1$ or $f \in \mathcal{S}_r$.

Theorem 2.2. *For $(r+1)/2 < \alpha < r/2 + 1$, f_α is differentiable at x_0 if and only if f is cuspidal at x_0 . In this case the value of the derivative is*

$$f'_\alpha(x_0) = \frac{(2\pi)^\alpha}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} f'(x_0 + it) dt.$$

Remark. In fact we shall prove the stronger result that

$$f_\alpha(x_0 + h) - f_\alpha(x_0) = h \frac{(2\pi)^\alpha}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} f'(x_0 + it) dt + O(h^{2\alpha-r} + h^2)$$

when f is cuspidal at x_0 , which suggests that the range $(r+1)/2 < \alpha < r/2 + 1$ is the natural one on which to study differentiability (see the remark after Corollary 2.1.1 and note that, in fact, by Lemma 3.2 a), f_α is everywhere differentiable for $\alpha > r/2 + 1$ if $f \in \mathcal{S}_r$).

Corollary 2.2.1. *For $(r+1)/2 < \alpha < r/2 + 1$ and $f \in \mathcal{S}_r$, f_α is differentiable in \mathbf{Q} and non-differentiable in $\mathbf{R} - \mathbf{Q}$.*

For $f = \sum a_n e(nz) \in \mathcal{S}_r$, the L -function $L(f, s)$ is defined to be the entire continuation of $\sum a_n n^{-s}$ (see [11]). Given a primitive character, χ , for some groups and multiplier systems appearing in the classical setting (see §7.4 [10]), $f = \sum a_n e(nz) \in \mathcal{S}_r(\Gamma, m)$ implies that the twisted series $\sum a_n \overline{\chi}(n) e(nz) \in \mathcal{S}_r(\Gamma', m')$. If this is the case (as we assume in the second part of the next result), we denote the latter automorphic form by $f_{\overline{\chi}}$.

Corollary 2.2.2. *For $f \in \mathcal{S}_r$ and $\alpha > (r+1)/2$, the value of the derivative of f_α at the rationals satisfies*

$$f'_\alpha(0) = 2\pi i L(f, \alpha - 1)$$

and

$$\sum_{a=1}^b \chi(a) f'_\alpha\left(\frac{a}{b}\right) = 2\pi i \tau(\chi) L(f_{\overline{\chi}}, \alpha - 1),$$

where χ is a primitive character modulo b and $\tau(\chi)$ is its Gauss sum.

Corollary 2.2.3. *If $r < 1$, the value of $f'_1(x)$ is $-2\pi ia_0$ at every point where it exists.*

In the next two results we shall assume that f has real Fourier coefficients. In this case $\Re f_\alpha$ and $\Im f_\alpha$ are cosine and sine real Fourier series respectively. Let $g_\alpha : \mathbf{R} \rightarrow \mathbf{R}$ be either of them.

Corollary 2.2.4. *For $(r+1)/2 < \alpha < r/2 + 1$, g_α is differentiable at x_0 if and only if f is cuspidal at x_0 .*

Now we shall establish that the graph of g_α is a fractal set. The non-cuspidal case appears to be more difficult to treat, and we shall only cover the range $r \leq 1$.

Theorem 2.3. *Let \mathcal{G} be the graph of g_α . For $f \in \mathcal{M}_r - \mathcal{S}_r$ with $r \leq 1$ or $f \in \mathcal{S}_r$ (without restrictions on r)*

$$\dim(\mathcal{G}) = \max(1, 2 - \alpha + r/2).$$

The relevance of automorphic forms in modern number theory suggests several applications and reformulations of the previous theorems. We mention here the case of elliptic curves.

Eichler-Shimura theory assigns an automorphic form $f \in S_2(\Gamma_0(N))$ to any abelian variety over \mathbf{Q} appearing as a factor of the Jacobian of $X_0(N)$ (see §7.5 of [14]), in such a way that the coefficients of its Hasse-Weil L -function coincide with the (normalized) Fourier coefficients of f . If one of these abelian varieties is one-dimensional, then it is an elliptic curve and it is said to be modular. One of the major achievements of contemporary mathematics is the recently proved Shimura-Taniyama-Weil conjecture establishing that, in fact, every elliptic curve over \mathbf{Q} is modular. The L -function $L(f, s)$ corresponding to the automorphic form associated to an elliptic curve E/\mathbf{Q} , which coincides with its Hasse-Weil L -function, reflects some of its arithmetical properties. For instance, after the work of Kolyvagin, $L(f, 1) \neq 0$ implies that E has vanishing rank, i.e., E contains finitely many rational points.

After these comments, the previous results at once imply

Proposition 2.4. *Let E/\mathbf{Q} be a modular elliptic curve and $\sum a_n n^{-s}$ its Hasse-Weil L -function. Consider the real functions*

$$A_\alpha(x) = \sum \frac{a_n}{n^\alpha} \cos(2\pi n x), \quad B_\alpha(x) = \sum \frac{a_n}{n^\alpha} \sin(2\pi n x).$$

Then the following results hold: i) A_α and B_α are differentiable in \mathbf{Q} and non-differentiable in $\mathbf{R} - \mathbf{Q}$ for $3/2 < \alpha < 2$; ii) if $B'_2(0) \neq 0$ then E has vanishing rank; and iii) for $1 < \alpha < 2$, the graphs of A_α and B_α are fractal sets of dimension $3 - \alpha$.

In principle, the slow convergence of the arising series suggests that pictures of these fractal graphs might be difficult to approximate with a personal computer; but using the PARI package they can be drawn in a reasonable amount of time. In §6 we shall check that for elliptic curves of low conductor it is possible to observe in these pictures an $x^{1+\delta} \sin(x^{-1})$ -like oscillation (the so-called “trigonometric chirps”) at differentiability points with small denominator.

To finish this section we illustrate our results, applying them to “Riemann’s example”

$$F(x) = \sum_{n=1}^{\infty} \frac{\sin(2\pi n^2 x)}{n^2},$$

obtaining results that had been established earlier by several authors.

Note that $F(x) = \frac{1}{2}\Im\theta_1(x)$, where $\theta(z) = \sum_{n=-\infty}^{\infty} e(n^2 z) \in \mathcal{M}_{1/2}(\Gamma_0(4), m_\theta)$.

The non-equivalent cusps of $\Gamma_0(4)\backslash\mathbb{H}$ are $i\infty$, 0 and $1/2$. As θ is not cuspidal at $i\infty$ or 0 and cuspidal at $1/2$, Corollary 2.2.4 (with $\alpha = 1$, $r = 1/2$) implies that $F(x)$ is differentiable in the orbit of $1/2 = \{a/(2b) : 2 \nmid a, b\}$ [6] and non-differentiable elsewhere [7]. By Corollary 2.2.3, the value of the derivative (when it exists) is $-\pi$ [6], and by Theorem 2.3 the fractal dimension of its graph [3] is $5/4$.

§3. SOME AUXILIARY RESULTS

In this section we shall state and prove some results that will be used in the next section. Here and hereafter we shall refer to the identity

$$\int_0^\infty t^{\alpha-1} e^{-2\pi n t} dt = \frac{\Gamma(\alpha)}{(2\pi n)^\alpha}$$

as the “gamma integral”, and it will be a basic tool to introduce the weight $n^{-\alpha}$ in the coefficients of an automorphic form.

Our first result determines the ranges in which the definition of f_α makes sense.

Proposition 3.1. *Let σ_α be the series defining f_α .*

- a) *If $f \in \mathcal{S}_r - \{0\}$, then σ_α converges to a continuous function for $\alpha > r/2$ and diverges in a dense set for $\alpha \leq r/2$.*
- b) *If $f \in \mathcal{M}_r - \mathcal{S}_r$, then σ_α converges to a continuous function for $\alpha > r$ and diverges in a dense set for $\alpha \leq r$.*

The main ingredient of the proof is the following result. For convenience, a few necessary but straightforward consequences of the expansion at the cusps (Lemma 3.3 and Lemma 3.5) are stated later.

Lemma 3.2 (cf. [10]). *Let $f(z) = \sum a_n e(nz) \in \mathcal{M}_r - \{0\}$. For C_1, C_2 depending only on f and every $N \geq 2$,*

- a) *if $f \in \mathcal{S}_r$, then $\sum_{n \leq N} a_n e(nx) = O(N^{r/2} \log N)$ uniformly in x ;*
- b) *if $f \in \mathcal{M}_r - \mathcal{S}_r$, then $\sum_{n \leq N} a_n e(nx) = O(N^r \log N)$ uniformly in x ;*
- c) *if $f \in \mathcal{S}_r$, then $C_1 N^r < \sum_{n \leq N} |a_n|^2 < C_2 N^r$;*
- d) *if $f \in \mathcal{M}_r - \mathcal{S}_r$, then $C_1 \phi_r(N) < \sum_{n \leq N} |a_n|^2 < C_2 \phi_r(N)$, where*

$$\phi_r(N) = \begin{cases} N^r & \text{if } 0 < r < 1, \\ N \log N & \text{if } r = 1, \\ N^{2r-1} & \text{if } r > 1. \end{cases}$$

Proof. a) (Th. 5.3 of [10]). The function $g(z) = |\Im z|^{r/2}|f(z)|$ is well defined in $\Gamma \backslash \mathbb{H}$, i.e., $g(\gamma z) = g(z)$ for $\gamma \in \Gamma$, and tends to zero at the cusps; hence g is bounded. Let $D_N(z) = \sum_{|n| \leq N} e(nz)$, the Dirichlet kernel. Then

$$\left| \sum_{n \leq N} a_n e(nx) \right| \leq \int_0^1 N^{r/2} g(u + i/N) |D_N(x - u - i/N)| du \leq C N^{r/2} \log N.$$

b) The proof is like that of a) but using $|f(u + i/N)| < C N^r$, which is a consequence of the expansion at the cusps after Farey subdivision of level N (see Lemma 3.5 below and [8]).

c) Let g be the function defined in a). Consider the intervals $|x - a/b| \leq 1/(bN^{1/2})$, with $C_1 N^{1/2} < b < C_2 N^{1/2}$; by the expansion at the cusp a/b , there exists a constant $C > 0$ such that $|\{x : g(x + i/N) > C\}| > 0$. As g is also bounded from above, by Parseval's identity

$$K_1 N^r < N^r \int_0^1 |g(u + i/N)|^2 du = \sum_n |a_n|^2 e^{-4\pi n/N} < K_2 N^r.$$

The upper bound implies at once

$$(3.1) \quad \sum_{n \leq N} |a_n|^2 < C_2 N^r.$$

On the other hand,

$$\sum_{n \leq KN} |a_n|^2 \geq \sum_n |a_n|^2 e^{-4\pi n/N} - \sum_{n > KN} |a_n|^2 e^{-4\pi n/N},$$

which combined with the lower bound and (3.1) gives

$$\sum_{n \leq KN} |a_n|^2 \geq K_1 N^r - C_2 e^{-4\pi K} N^r.$$

Choosing K large enough, we get the result. (A more precise estimate can be obtained with less elementary arguments involving Rankin-Selberg convolution.)

d) The interval $[0, 1]$ is covered by the union of

$$J(A/B, k) = \left\{ x : \frac{k-1}{N} \leq |x - \frac{A}{B}| \leq \frac{k}{N} \right\},$$

where $1 \leq k \leq N^{1/2}/B$ and A/B is irreducible, $0 \leq A \leq B \leq N^{1/2}$.

For each $x \in J(A/B, k)$ with $k > 1$, let a/b be the nearest Farey fraction of level $2N/(kB)$. As $|x - A/B| \geq (k-1)/N$, by well known approximation properties (see Th. 36 of [8]), we have $|x - a/b| < kB/(2Nb) \leq b^{-2}$ and $a/b \neq A/B$, which combined with $|x - A/B| \leq k/N$ and $|a/b - A/B| \geq 1/(bB)$ gives $b \geq CN/(kB)$. Hence Lemma 3.5 implies that

$$|f(x + i/N)| \leq C(B^r k^r + N^r B^{-r} k^{-r}) \quad \text{for } x \in J(A/B, k),$$

and we obtain

$$(3.2) \quad \int_0^1 |f(x + i/N)|^2 dx \leq C \sum_{A, B, k} (B^{2r} k^{2r} N^{-1} + N^{2r-1} B^{-2r} k^{-2r}) \leq K_2 \phi_r(N).$$

For the lower bound, note that, as we mentioned in c), if

$$g(x + i/N) = N^{-r/2} |f(x + i/N)| > C$$

on a set of positive measure, then

$$(3.3) \quad K_1 \phi_r(N) \leq \int_0^1 |f(x + i/N)|^2 dx \quad \text{for } 0 < r < 1.$$

The right hand side also is greater than

$$\sum_{A,B} \int_{J(A/B,1)} |f(x + i/N)|^2 dx \geq C \sum_{A,B} N^{2r-1} B^{-2r}.$$

Here we have used the expansion at the cusps (in the form of Lemma 3.3), and the summation is restricted to $0 \leq A \leq B \leq C' N^{1/2}$, with A/B irreducible and equivalent to a cusp in which f is not cuspidal. As $[SL_2(\mathbf{Z}) : \Gamma] < \infty$, the latter condition can be ruled out, keeping a comparable value of the sum (there are finitely many non-equivalent cusps A/B). Hence we have proved

$$(3.4) \quad \int_0^1 |f(x + i/N)|^2 dx \geq K_1 \phi_r(N) \quad \text{for } r \geq 1.$$

We deduce from (3.2), (3.3) and (3.4) that

$$K_1 \phi_r(N) \leq \sum_n |a_n|^2 e^{-4\pi n/N} \leq K_2 \phi_r(N),$$

and the proof is completed as in c). \square

Our next result is a simple consequence of the expansion of f at a cusp.

Lemma 3.3. *Let $f \in \mathcal{M}_r$ and $z = x + iy \in \mathbb{H}$. If $|bz - a|^2/y < \delta$ for a, b relatively prime integers, then*

$$|f(z)| < C(\delta) |bz - a|^{-r} e^{-Ky|bz-a|^{-2}},$$

and if $|bz - a|^2/y \rightarrow 0$, then

$$|f(z)| |bz - a|^r e^{Ky|bz-a|^{-2}} \rightarrow C_1,$$

where $C_1 > 0$ and $K \geq 0$ depend only on the equivalence class of the cusp $\mathfrak{a} = a/b$, and $K = 0$ if and only if f is not cuspidal at \mathfrak{a} .

Proof. Let $\sigma_{\mathfrak{a}}$ be the scaling matrix for \mathfrak{a} ; then $\sigma_{\mathfrak{a}}^{-1}(\mathfrak{a}) = \infty$ and we can write

$$\sigma_{\mathfrak{a}}^{-1} = \begin{pmatrix} C & 0 \\ 0 & 1/C \end{pmatrix} \gamma \quad \text{with } \gamma = \begin{pmatrix} * & * \\ b & -a \end{pmatrix} \in SL_2(\mathbf{Z})$$

and $C \neq 0$ depending only on the equivalence class of \mathfrak{a} . Hence

$$\Im \sigma_{\mathfrak{a}}^{-1}(z) = \frac{\Im z}{|j_{\sigma_{\mathfrak{a}}^{-1}}(z)|^2} = \frac{C^2 y}{|bz - a|^2},$$

and the lemma follows from the expansion of f at the cusp \mathfrak{a} (see §2) upon replacing z by $\sigma_{\mathfrak{a}}^{-1}(z)$. (Note that $j_{\sigma_{\mathfrak{a}}}(\sigma_{\mathfrak{a}}^{-1}(z)) = (j_{\sigma_{\mathfrak{a}}^{-1}}(z))^{-1}$). \square

In the following two lemmas we shall use generalised Ford circles (Speiser circles)

$$\left\{ z : \left| z - \frac{a}{c} - \frac{i}{tc^2} \right| \leq \frac{1}{tc^2} \right\} = \gamma(\left\{ \Im z \geq \frac{t}{2} \right\}), \quad \gamma = \begin{pmatrix} a & * \\ c & * \end{pmatrix} \in SL_2(\mathbf{Z}), \quad c \neq 0, \quad t > 0.$$

Let us denote each of these circles by $\mathcal{F}_{a/c}$. They have the following elementary properties: For $t = 1$, $\bigcup \mathcal{F}_{a/c} \supset \{0 < \Im z < 1/2\}$ because the standard fundamental domain of $SL_2(\mathbf{Z}) \backslash \mathbb{H}$ is contained in $\{\Im z \geq 1/2\}$. For $t = 2$ (Ford circles) they form

a set of mutually tangent circles in such a way that $\mathcal{F}_{a/c}$ is tangent to $\mathcal{F}_{a'/c'}$ if and only if a/c and a'/c' are consecutive Farey functions. For $t = \sqrt{5}$, Hurwitz' theorem in diophantine approximation (see Th. 193 of [8]) assures that the projections of the $\mathcal{F}_{a/c}$ on the real axis cover each irrational number infinitely many times; in fact, Hurwitz' theorem itself can be proved using the geometry of Ford circles (see [5]).

In general, there are striking relations between this and other generalizations of Ford circles, geodesics in Riemann surfaces and results in diophantine approximations (see [12]).

Lemma 3.4. *Let $f \in \mathcal{M}_r - \{0\}$ and $x_0 \in \mathbf{R}$. Then*

$$\overline{\lim_{y \rightarrow 0^+}} y^{r/2} |f(x_0 + iy)| \neq 0 \Leftrightarrow f \text{ is not cuspidal at } x_0,$$

and this upper limit is finite everywhere for $f \in \mathcal{S}_r$.

Proof. The last part of the lemma is just a consequence of the properties of the function g , introduced in the proof of Lemma 3.2 a), and the first part is trivial for $x_0 \in \mathbf{Q}$ because of the expansion at the cusps (see Lemma 3.3). Hence it remains only to prove that the upper limit does not vanish for $x_0 \notin \mathbf{Q}$.

Let $\gamma_1^{-1}, \gamma_2^{-1}, \dots$ be representatives of $SL_2(\mathbf{Z})/\Gamma$. Then a fundamental domain \mathcal{D} of $\Gamma \setminus \mathbb{H}$ can be constructed applying the γ_i 's to the standard one of $SL_2(\mathbf{Z}) \setminus \mathbb{H}$. As the number of zeros of f in \mathcal{D} is finite, we can choose $2 < t_0 < \sqrt{5}$ such that $|f(z)| \geq C > 0$ for $z \in \mathcal{D} \cap \bigcup \gamma_i(\{\Im z = t_0/2\})$. As the line $\{\Re z = x_0\}$ cuts infinitely many of the generalised Ford circles for $t \leq \sqrt{5}$, we can find $z = x_0 + iy$ with y arbitrarily small such that $z = \gamma(\sigma_0 + it_0/2)$ with $|\sigma_0| \leq 1/2$, $\gamma \in SL_2(\mathbf{Z})$. Writing $\gamma^{-1} = \gamma_i^{-1} \tilde{\gamma}$ with $\tilde{\gamma} \in \Gamma$, we get

$$\begin{aligned} |f(x_0 + iy)| &= |f(\gamma_i(\sigma_0 + it_0/2))| |j_{\gamma_i \gamma^{-1}}(z)|^{-r} \\ &\geq C |j_{\gamma_i}(\sigma_0 + it_0/2)|^{-r} |j_{\gamma^{-1}}(z)|^{-r} \\ &\geq C' |j_{\gamma^{-1}}(z)|^{-r} = C' t_0^{r/2} (2y)^{-r/2}, \quad C' > 0, \end{aligned}$$

and the result follows. \square

Lemma 3.5. *If $|x - a/b| \leq b^{-2}$ with a/b an irreducible fraction, then*

$$|f(x + iy)| < C(b^{-r}y^{-r} + b^r).$$

Proof. Consider Ford circles for $t = 1$. Let $\mathcal{F}_{A/B}$ be the circle containing $z = x + iy$ (we assume $0 < y < 1/2$). Note that

$$\left| z - \frac{A}{B} - \frac{i}{B^2} \right| \leq \frac{1}{B^2} \Rightarrow y \leq \frac{2}{B^2}, \quad \left| x - \frac{A}{B} \right|^2 \leq 2y \Rightarrow \frac{|Bz - A|^2}{y} \leq 4,$$

and by Lemma 3.3 we obtain

$$(3.5) \quad |f(x + iy)| \leq K((Bx - A)^2 + B^2 y^2)^{-r/2}.$$

If $B < b/2$, then

$$\left| x - \frac{A}{B} \right| \geq \left| \frac{A}{B} - \frac{a}{b} \right| - \left| x - \frac{a}{b} \right| \geq \frac{1}{bB} - \frac{1}{b^2} > \frac{1}{2bB}$$

and (3.5) gives $|f(x + iy)| \leq K(2b)^r$. On the other hand, if $B \geq b/2$ we obtain $|f(x + iy)| \leq KB^{-r}y^{-r} \leq K(by/2)^{-r}$ from (3.5). \square

After these lemmas, Proposition 3.1 follows by standard arguments.

Proof (of Proposition 3.1). By Lemma 3.2 a), b) and partial summation it is plain that σ_α converges for $\alpha > r/2$ when $f \in \mathcal{S}_r$ and for $\alpha > r$ when $f \in \mathcal{M}_r - \mathcal{S}_r$. It is also easy to prove that the function obtained is continuous (this can be deduced from Th. XII.9.1 of [17]).

Let us prove first that $\sigma_{r/2}(x)$ diverges almost everywhere. Considering the kernels of summability $\varphi_1(u) = e^{-2\pi u}(u^{r/2} + 1)$ and $\varphi_2(u) = e^{-2\pi u}$, we have (see Th. III.1.2 of [17])

$$\lim_{t \rightarrow 0^+} t^{r/2} f(x + it) = \lim_{t \rightarrow 0^+} \left(\sum \frac{a_n}{n^{r/2}} e(nx) \varphi_1(nt) - \sum \frac{a_n}{n^{r/2}} e(nx) \varphi_2(nt) \right) = 0$$

if $\sigma_{r/2}(x)$ converges; but Lemma 3.4 (see above) implies that this only holds in, at most, a countable set of values of x .

Finally we shall prove that $\sigma_r(x)$ does not converge at every cusp x_0 at which f is not cuspidal. If $\sigma_r(x_0)$ converges to ℓ , by the gamma integral,

$$\ell = \lim_{u \rightarrow 0^+} \sum \frac{a_n}{n^r} e(nx_0) e^{-2\pi nu} = \frac{(2\pi)^r}{\Gamma(r)} \lim_{u \rightarrow 0^+} \int_u^\infty (t-u)^{r-1} (f(x_0 + it) - a_0) dt.$$

But the expansion at the cusp

$$f(x_0 + it) = C_1 t^{-r} + O(e^{-C_2/t})$$

and the divergence of the integral

$$\int_u^1 (t-u)^{r-1} t^{-r} dt$$

as $u \rightarrow 0^+$ imply $\ell = \infty$. □

Finally, we present some lemmas of an analytic nature.

Lemma 3.6. *Let g be a 1-periodic continuous function and let $G_r = G_r(x)$ be its Poisson integral. If $g \in \lambda_\beta(x_0)$, $0 < \beta < 1$, then for any positive integer m ,*

$$\lim_{r \rightarrow 1^-} (1-r)^{m-\beta} G_r^{(m)}(x_0) = 0.$$

Proof. The result is a simple generalization of Lemma 2.11 of [7] that can be obtained from the proof given there by just replacing the estimates for $P'_r(t)$ by those for $P_r^{(m)}(t)$ (see (3.9) below). □

Lemma 3.7. *For $g : [0, 1] \rightarrow \mathbf{R}$ continuous and non-constant, let \mathcal{G} be its graph and define $G : [0, 1] \rightarrow \mathbf{R}$ as*

$$G(x) = \int_0^{1-x} |g(x+t) - g(t)| dt.$$

a) *If $g \in \Lambda_\beta$ for every $\beta < \beta_0$, then*

$$\overline{\dim}(\mathcal{G}) \leq 2 - \beta_0.$$

b) *If $\varliminf_{x \rightarrow 0^+} G(x)x^{-\beta} = \infty$ for every $\beta > \beta_1$, then*

$$\underline{\dim}(\mathcal{G}) \geq 2 - \beta_1.$$

Proof. The proof of a) is well known (see, for instance, Th. 8.1 of [4]). For b) note first that

$$\sup_{x,y \in [jh, (j+2)h]} |g(x) - g(y)| \geq \frac{1}{h} \int_{jh}^{(j+1)h} |g(x+h) - g(x)| dx,$$

and the supremum is less than or equal to the sum of the suprema over $[jh, (j+1)h]$ and $[(j+1)h, (j+2)h]$. Hence, substituting in the definition of $\dim(\mathcal{G})$, we get

$$\underline{\dim}(\mathcal{G}) \geq \lim_{h \rightarrow 0^+} \frac{\log(h^{-2}G(h))}{\log h} \geq 2 - \beta,$$

and the result follows upon letting $\beta \rightarrow \beta_1^+$. \square

The next results are sharper tools to calculate the dimension.

Proposition 3.8. *Let g be a continuous function, $g : [0, 1] \rightarrow \mathbf{R}$ and $p \geq 1$. If $g = \sum g_n$ with $g_n \in W^{2,p}$, then*

$$V(g, h) \leq C \sum_n \min(T_p(g'_n), h^{-1}T_p(g_n)),$$

where $T_p(u) = \|u\|_p + h^{1/p} \|u\|_p^{1/p'} \|u'\|_p^{1/p}$, $1/p + 1/p' = 1$.

Proof. By density (Th. VIII.6 [1]) we can assume that g_n is differentiable. By the mean value theorem we have

$$V(g_n, h) = h \sum_j |g'_n(\xi_j)|, \quad (j-1)h < \xi_j < jh.$$

Sobolev's inequality (cf. (29) on p. 168 of [1]) applied to $(g'_n)^p$ implies that

$$|g'_n(\xi_j)|^p \leq \frac{1}{|I|} \int_I |g'_n|^p + p \int_I |g'_n|^{p-1} |g''_n|$$

for any interval $I \subset [0, 1]$. Choosing $I = I_j = [(j-1)h, jh]$, summing over j and applying Hölder's inequality, we obtain

$$|g'_n(\xi_j)|^p \leq h^{-1} \|g'_n\|_p^p + p \|g'_n\|_p^{p-1} \|g''_n\|_p \leq Ch^{-1} |T_p(g'_n)|^p.$$

Hence

$$(3.6) \quad V(g_n, h) \leq (h \sum_j |g'_n(\xi_j)|^p)^{1/p} \leq C' T_p(g'_n).$$

Trivially we have

$$V(g_n, h) \leq 2 \sum_j |g_n(\xi'_j)|, \quad \xi'_j \in I_j.$$

The same argument as before proves

$$(3.7) \quad V(g_n, h) \leq 2h^{-1} (h \sum_j |g_n(\xi'_j)|^p)^{1/p} \leq C'' T_p(g_n),$$

and the result follows from (3.6) and (3.7) upon summing over n . \square

Corollary 3.9. *Let $g : [0, 1] \rightarrow \mathbf{R}$ be continuous and let \mathcal{G} be its graph. If the Fourier coefficients, c_k , of g satisfy*

$$\sum_{N \leq |k| < 2N} |c_k|^2 = O(N^{-\beta}), \quad \text{with } \beta > 1,$$

then

$$\overline{\dim}(\mathcal{G}) \leq \max(1, 2 - \beta/2).$$

Remark. Note that this result can be employed to obtain the upper bound in Theorem 3.1 of [3].

Proof. Simply choose $g_0 \equiv c_0$ and

$$g_N(x) = \sum_{N \leq |k| < 2N} c_k e(kx) \text{ with } N = 2^n.$$

Parseval's identity implies that

$$T_2(g'_N) = O(N^{1-\beta/2} + h^{1/2} N^{3/2-\beta/2})$$

and

$$T_2(g_N) = O(N^{-\beta/2} + h^{1/2} N^{1/2-\beta/2}).$$

The result is now a consequence of Proposition 3.8 (using the first bound for $1 \leq N \leq h^{-1}$ and the second one for $N > h^{-1}$). \square

Proposition 3.10. *Let g be a 1-periodic continuous real function, \mathcal{G} its graph (restricted to $[0, 1]$) and $G_r(x)$ its Poisson integral. If for each $h \in (0, 1)$ there exist $x_1, x_2, \dots, x_N \in [0, 1]$ with $x_{j+1} - x_j \geq h$ and $N > C_1 h^{-\beta_1} > 1$ such that*

$$h^{m-\beta_2} |G_{1-h}^{(m)}(x_j)| > C_2 > 0, \quad j = 1, 2, \dots, N,$$

for some positive integer m , then

$$\underline{\dim}(\mathcal{G}) \geq \max(1, 1 + \beta_1 - \beta_2).$$

Proof. By the Poisson integral representation we get

$$(3.8) \quad \left| \int_{x_j-1/2}^{x_j+1/2} (g(x) - g(x_j)) P_{1-h}^{(m)}(x_j - x) dx \right| > C_2 h^{-m+\beta_2},$$

where $P_r(x) = \sum r^{|n|} e(nx)$ is the Poisson kernel (note that $\int P_r^{(m)} = 0$).

The explicit formula for P_r implies the bounds

$$(3.9) \quad \begin{aligned} |P_r^{(m)}(x)| &= O((1-r)x^{-m-2}) & \text{for } |x| > 1-r, \\ |P_r^{(m)}(x)| &= O((1-r)^{-m-1}) & \text{for } |x| \leq 1-r. \end{aligned}$$

Consider the intervals

$$J_{-1}(x_j) = \emptyset, \quad J_k(x_j) = (-2^k h + x_j, x_j + 2^k h), \quad k = 0, 1, 2, \dots$$

Dyadic subdivision in (3.8) implies that we can find k_j and $y_j \in J_{k_j}(x_j) - J_{k_j-1}(x_j)$ such that

$$2^{k_j} h |g(y_j) - g(x_j)| |P_{1-h}^{(m)}(x_j - y_j)| > C_3 h^{-m+\beta_2+\varepsilon}$$

for any $\varepsilon > 0$.

For the sake of brevity we shall write T_j instead of $J_{k_j}(x_j)$. Using (3.9), the last inequality gives

$$|T_j| \cdot |g(y_j) - g(x_j)| \cdot h |T_j|^{-m-2} > K_\varepsilon h^{-m+\beta_2+\varepsilon}.$$

In particular, using $|T_j|^{-m} < h^{-m}$, we get

$$(3.10) \quad V(g, h; T_j) > K_\varepsilon h^{\beta_2-1+\varepsilon} |T_j|.$$

As $T_j \supset (x_j - h/2, x_j + h/2)$,

$$|\bigcup_j T_j| \geq Nh > C_1 h^{1-\beta_1}.$$

By a covering lemma (see Lemma 7.3 of [13]) we can choose a subset $\{T_{j_n}\} \subset \{T_j\}$ such that the T_{j_n} 's are disjoint and satisfy

$$|\bigcup_n T_{j_n}| = \sum_n |T_{j_n}| \geq C'_1 h^{1-\beta_1}.$$

Summing in (3.10) over $j = j_n$, we finally obtain

$$V(g, h) > K'_\varepsilon h^{\beta_2-\beta_1+\varepsilon}$$

for any $\varepsilon > 0$, and the lower bound for the dimension follows. \square

§4. PROOF OF THE MAIN RESULTS

Proof of Theorem 2.1. If f is a cusp form, it is easy to deduce from the estimates for its Fourier coefficients that $f_\alpha \in \Lambda_{\alpha-r/2}^{\log}$, $0 < \alpha - r/2 < 1$. Namely, given $0 < h < 1$ and $N > 1$, we can write

$$\begin{aligned} |f_\alpha(x+h) - f_\alpha(x)| &\leq \left| \sum_{n < N} \frac{a_n}{n^\alpha} e(nx)(e(nh) - 1) \right| + \left| \sum_{n \geq N} \frac{a_n}{n^\alpha} (e(n(x+h)) - e(nx)) \right| \\ &= |S_1| + |S_2|. \end{aligned}$$

Applying the mean value theorem to the real and imaginary parts of S_1 , we have

$$|S_1| \leq 4\pi h \left| \sum_{n < N} \frac{a_n}{n^{\alpha-1}} e(n(x+\xi)) \right|$$

for some $\xi \in (0, h)$. Using the estimate of the proof of Lemma 3.2 a) and applying partial summation to the latter inequality and to S_2 , we find that

$$|f_\alpha(x+h) - f_\alpha(x)| \leq C(hN^{r/2-\alpha+1} + N^{r/2-\alpha}) \log N.$$

Finally, choosing $N = h^{-1}$, we get that $f_\alpha \in \Lambda_{\alpha-r/2}^{\log}$.

Note that it remains only to prove that $f_\alpha \notin \lambda_{\alpha-r/2}(x_0)$ if f is not cuspidal at x_0 . Our arguments can be considered an extension of [7].

Let $m = [\alpha] + 1$, where $[\cdot]$ indicates the integral part, and let

$$F(z) = \sum a_n n^{m-\alpha} e(nz).$$

If $f_\alpha \in \lambda_{\alpha-r/2}(x_0)$, by Lemma 3.6 with $r = e^{-2\pi y}$ we have

$$\lim_{y \rightarrow 0^+} (1 - e^{-2\pi y})^{m-\alpha+r/2} F(x_0 + iy) = 0;$$

hence for $\varepsilon > 0$ there exists $\delta > 0$ such that

$$(4.1) \quad |F(x_0 + iy)| < \varepsilon y^{\alpha-m-r/2}$$

for $0 < y < \delta$.

Using the gamma integral, we get

$$y^{r/2} f(x_0 + iy) = a_0 y^{r/2} + \frac{(2\pi)^{m-\alpha}}{\Gamma(m-\alpha)} \int_0^\infty t^{m-\alpha-1} y^{r/2} F(x_0 + i(t+y)) dt.$$

By (4.1), if $y < \delta/2$, the contribution to the integral in the range $0 < t < \delta/2$ is bounded by $C\varepsilon$. Trivially, the contribution of $t \geq \delta/2$ is less than $C_\delta y^{r/2}$. Hence we have proved that

$$\overline{\lim}_{y \rightarrow 0^+} y^{r/2} |f(x_0 + iy)| < C\varepsilon,$$

which contradicts Lemma 3.4 upon choosing ε arbitrarily small. \square

Corollary 2.1.1 is a trivial consequence of the theorem because there are no irrational cusps.

Proof of Theorem 2.2. By Theorem 2.1, if f is not cuspidal at x_0 , then $f_\alpha \notin \lambda_{\alpha-r/2}(x_0)$; hence we have to consider only the case in which f is cuspidal at x_0 .

Let $0 < h < 1$ (the case with negative h is similar). By the gamma integral

$$f_\alpha(x_0 + h) - f_\alpha(x_0) = \frac{(2\pi)^\alpha}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} (f(x_0 + h + it) - f(x_0 + it)) dt = \frac{(2\pi)^\alpha}{\Gamma(\alpha)} I.$$

Let us divide the range of integration, obtaining three integrals over the intervals $[0, y_1]$, $[y_1, y_2]$, $[y_2, \infty)$ respectively. Applying the mean value theorem to the real and imaginary parts of the integrand of the last integral (regarded as a function of h) and choosing $y_1 = h^2$, $y_2 = h$, we obtain

$$(4.2) \quad I = I_1 + I_2 + h(\Re I_3(\xi_1) + i\Im I_3(\xi_2)),$$

where $0 < \xi_j < h$ and

$$I_1 = \int_0^{h^2} t^{\alpha-1} (f(x_0 + h + it) - f(x_0 + it)) dt, \quad I_2 = \int_{h^2}^h t^{\alpha-1} (f(x_0 + h + it) - f(x_0 + it)) dt,$$

$$I_3(\xi) = \int_h^\infty t^{\alpha-1} f'(x_0 + \xi + it) dt.$$

Note that the exponential decay at the cusp x_0 implies that the contribution of $f(x_0 + it)$ to I_1 and I_2 is negligible in comparison with h .

If $f \in S_r$, then by the second part of Lemma 3.4

$$\int_0^{h^2} t^{\alpha-1} |f(x_0 + h + it)| dt < C \int_0^{h^2} t^{\alpha-r/2-1} dt = O(h^{2\alpha-r}).$$

Let x_1, x_2 be such that x_0, x_1, x_2 are consecutive Farey fractions with $x_0 < x_1 < x_0 + h \leq x_2$; then for $a/b = x_1$ or x_2 we have $|x_0 + h - a/b| \leq b^{-2}$. Since $x_1 - x_0 < h$ and $x_2 - x_0 \geq h$, it is easy to deduce (use elementary properties of Farey fractions [8]) that $b = O(h^{-1})$ and $h^{-1} = O(b)$, with an O -constant depending on the denominator of x_0 . Hence by Lemma 3.5, for $f \in \mathcal{M}_r$,

$$\int_0^{h^2} t^{\alpha-1} |f(x_0 + h + it)| dt < C \int_0^{h^2} t^{\alpha-1} (h^r t^{-r} + h^{-r}) dt = O(h^{2\alpha-r}).$$

Therefore we have proved

$$(4.3) \quad I_1 = O(h^{2\alpha-r}).$$

Taking $z = x_0 + h + it$ and $a/b = x_0$ in Lemma 3.3, we obtain $|f(x_0 + h + it)| \leq C_1 h^{-r} e^{-C_2 t h^{-2}}$; hence we have also

$$(4.4) \quad I_2 = O(h^{2\alpha-r}).$$

Let $g(z) = -i(i(\xi - z))^{\alpha-1}f'(z+x_0)$ and let $\gamma_1, \gamma_2, \gamma_3$ be the straight lines forming the positively oriented border, ∂D , of $D = \{0 \leq \Re z \leq \xi, \Im z \geq h\}$ (with $\Re \gamma_1 = 0$ and $\Re \gamma_3 = \xi$). Then we can write $I_3(\xi)$ as

$$I_3(\xi) = \int_{\gamma_3} g(z)dz = - \int_{\gamma_1} - \int_{\gamma_2} + \int_{\partial D} = -J_1 - J_2 + J_3.$$

Note that J_3 vanishes identically because g is holomorphic in D (and decays at infinity). On the other hand, integrating by parts and using Lemma 3.3 as before, we get $|J_2| < C_1 h^{-r} e^{-C_2/h}$. Finally, using the Taylor expansion $(i(\xi - it))^{\alpha-1} = t^{\alpha-1} + O(\xi t^{\alpha-2})$, we deduce that

$$J_1 = - \int_h^\infty t^{\alpha-1} f'(x_0 + it) dt + O(h) = - \int_0^\infty t^{\alpha-1} f'(x_0 + it) dt + O(h),$$

and hence

$$(4.5) \quad I_3(\xi) = \int_0^\infty t^{\alpha-1} f'(x_0 + it) dt + O(h).$$

Substituting (4.3), (4.4) and (4.5) in (4.2), we obtain

$$f_\alpha(x_0 + h) - f_\alpha(x_0) = h \frac{(2\pi)^\alpha}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} f'(x_0 + it) dt + O(h^{2\alpha-r}) + O(h^2),$$

and the result follows. \square

Corollary 2.2.1 is an immediate consequence of Theorem 2.2 because, by definition, a cusp form is cuspidal exactly at every cusp.

Proof of Corollary 2.2.2. By the gamma integral, for $s \in \mathbf{R}^+$

$$L(f, s) = \frac{(2\pi)^s}{\Gamma(s)} \int_0^\infty t^{s-1} f(it) dt.$$

Note that the integral converges (because $f \in S_r$) and gives an analytic continuation of $L(f, s) = \sum a_n n^{-s}$ over the whole complex plane (defining $1/\Gamma(s)$ as zero for $s = 0, -1, -2, \dots$).

Integrating by parts in the formula for $f'_\alpha(x_0)$ and taking $x_0 = 0$, we get

$$(4.6) \quad f'_\alpha(0) = \frac{(2\pi)^\alpha}{\Gamma(\alpha-1)} i \int_0^\infty t^{\alpha-2} f(it) dt = 2\pi i L(f, \alpha-1).$$

By properties of Gauss sums,

$$\sum_{a=1}^b \chi(a) f(z + a/b) = \tau(\chi) g(z),$$

where $g = f_{\overline{\chi}}$. Hence

$$\sum_{a=1}^b \chi(a) f'_\alpha(a/b) = \tau(\chi) g'_\alpha(0),$$

and the result is a consequence of (4.6). \square

Corollary 2.2.3 is a particular case of the formula for $f'_\alpha(x_0)$ because $f(x_0 + it) \rightarrow 0$ as $t \rightarrow 0^+$ and $f(x_0 + it) \rightarrow a_0$ as $t \rightarrow +\infty$.

Proof of Corollary 2.2.4. Assume that $g_\alpha = \Re f_\alpha$, the case $g_\alpha = \Im f_\alpha$ being similar.

Trivially, by Theorem 2.2, if f is cuspidal at x_0 , then g_α is differentiable at x_0 . On the other hand, if g_α is differentiable at x_0 , then $g_\alpha \in \Lambda_\beta(x_0)$ for $\alpha - r/2 < \beta < 1$, and the boundness of the conjugate operator in Lipschitz spaces (see the following remark and III.13.29 of [17]) implies that $\Im f_\alpha \in \Lambda_\beta^{\log}(x_0)$; hence $f_\alpha \in \lambda_{\alpha-r/2}(x_0)$, and f is cuspidal at x_0 by Theorem 2.1. \square

Remark. Note that III.13.29 of [17] is stated in global form but, with the notation used there, the proof (via 13.30) can be extended to $f \in \Lambda_\beta(x_0) \cap \Lambda_\varepsilon$ simply by employing the estimate

$$\int_{-2h}^{2h} \frac{|f(x+t) - f(x+h)|}{|t-h|} dt \leq C \int_{-2h}^{2h} \frac{\min(|t-h|^\varepsilon, |t|^\beta + h^\beta)}{|t-h|} dt = O(h^\beta |\log h|)$$

to bound the corresponding part of the integral defining $\tilde{f}(x+h)$.

In proving Theorem 2.3 we shall distinguish the cuspidal and the non-cuspidal cases because, as we mentioned before, the latter one requires finer considerations.

In both cases we shall assume in the proofs that $g_\alpha = \Re f_\alpha$. The case $g_\alpha = \Im f_\alpha$ is completely similar.

Proof of Theorem 2.3 (cuspidal case). By Theorem 2.1 we can choose $\beta_0 = \alpha - r/2$ in Lemma 3.7 a), obtaining, for $\alpha - r/2 < 1$,

$$\overline{\dim}(\mathcal{G}) \leq 2 + r/2 - \alpha.$$

As g_α is 1-periodic and continuous, with the notation of Lemma 3.7, for $g = g_\alpha$

$$G(x) = \int_0^1 |g(x+t) - g(t)| dt + O(x).$$

We have the trivial inequality

$$(4.8) \quad \int_0^1 |g(x+t) - g(t)| dt \geq \frac{\int_0^1 |g(x+t) - g(t)|^2 dt}{\sup_t |g(x+t) - g(t)|}.$$

By Parseval's identity the numerator of the right hand side is

$$4 \int_0^1 \left| \sum \frac{a_n}{n^\alpha} \sin(\pi n x) \sin(2\pi n(t+x/2)) \right|^2 dt = 2 \sum \frac{|a_n|^2}{n^{2\alpha}} \sin^2(\pi n x).$$

Hence, considering the contribution of the range $1 < 4nx < 3$ and using Lemma 3.2 c), we obtain

$$\int_0^1 |g(x+t) - g(t)|^2 dt > Cx^{2\alpha-r}.$$

On the other hand, by the mean value theorem the denominator in (4.8) can be written as

$$\left| -2\pi x \sum_{n < x^{-1}} \frac{a_n}{n^{\alpha-1}} \sin(2\pi(\xi + t_0)) + \sum_{n \geq x^{-1}} \frac{a_n}{n^\alpha} (\cos(2\pi n(x+t_0)) - \cos(2\pi n t_0)) \right|,$$

where $0 < \xi < x$ and t_0 is the value at which the supremum is achieved. Then, by Lemma 3.2 a), for $\alpha - r/2 < 1$

$$(4.9) \quad \sup_t |g(x+t) - g(t)| = O(x^{\alpha-r/2-\varepsilon}), \quad \forall \varepsilon > 0.$$

Substituting these bounds in (4.8) and (4.7), we conclude by Lemma 3.7 b) that

$$\underline{\dim}(\mathcal{G}) \geq 2 + r/2 - \alpha;$$

hence

$$\dim(\mathcal{G}) = 2 + r/2 - \alpha, \quad \text{for } \alpha - r/2 < 1.$$

Note that for $\alpha - r/2 \geq 1$, (4.9) holds upon replacing $\alpha - r/2 - \varepsilon$ by $1 - \varepsilon$, which immediately gives $\dim(\mathcal{G}) = 1$ by Lemma 3.7 a). \square

Proof of Theorem 2.3 (non-cuspidal case). a) The upper bound:

For $\alpha > 1$, the upper bound

$$\overline{\dim}(\mathcal{G}) \leq \max(1, 2 + r/2 - \alpha)$$

is a direct consequence of Corollary 3.9 and Lemma 3.2 d).

For $\alpha \leq 1$, we start by considering the (Dirichlet) covering of $[0, 1]$ given by the intervals

$$I_{a/b} = \left\{ x : \left| x - \frac{a}{b} \right| \leq \frac{h^{1/2}}{b} \right\},$$

where a/b is an irreducible fraction with $0 \leq a \leq b \leq h^{-1/2}$.

Applying the mean value theorem as in (4.2), we have for $x, y \in I_{a/b}$ that

$$|g_\alpha(x) - g_\alpha(y)| \leq C(|J_1| + h^{1/2}b^{-1}|J_2|),$$

where

$$J_1 = \int_0^{h^{1/2}b^{-1}} t^{\alpha-1} (f(x+it) - f(y+it)) dt, \quad J_2 = \int_{h^{1/2}b^{-1}}^\infty t^{\alpha-1} f'(\xi+it) dt$$

for some $\xi \in I_{a/b}$.

Lemma 3.5 applied to J_1 and J_2 , after partial integration, gives

$$|g_\alpha(x) - g_\alpha(y)| = O(h^{(\alpha-r)/2}b^{-\alpha} + h^{\alpha/2}b^{r-\alpha}).$$

Hence

$$\begin{aligned} V(g_\alpha, h) &\leq \sum_{a,b} V(g_\alpha, h; I_{a/b}) \\ &= O\left(\sum_{a,b} (h^{(\alpha-r-1)/2}b^{-\alpha-1} + h^{(\alpha-1)/2}b^{r-\alpha-1})\right) = O(h^{\alpha-r/2-1} \log h). \end{aligned}$$

This proves the upper bound for $\alpha \leq 1$.

b) The lower bound:

Define $y = -\frac{1}{2\pi} \log(1-h)$. Then $G_r(t)$, the Poisson integral of g_α , satisfies

$$G_{1-h}(x) = \Re f_\alpha(x + iy)$$

(where f_α is extended to the upper half plane in the obvious way).

We shall apply Proposition 3.10 with $x_j = a_j/b_j + \tau_j y$, where $0 \leq \tau_j < 1$ is to be chosen later and a_j/b_j are reduced fractions, with $y^{-1/2+\varepsilon} \leq b_j \leq C'y^{-1/2+\varepsilon}$, $0 < \varepsilon < 1/2$, at which f is not cuspidal. It is plain that for h small enough the fractions satisfy the spacing condition required in the lemma (because $b_j^{-2} \geq C'y^{1-2\varepsilon}$, $2\pi y \sim h$), and we can take $\beta_1 = 1 - 2\varepsilon$ (see the comments before (3.4) in the proof of Lemma 3.2).

Assume first that α is integral and even (the odd case proceeds in the same way, but with \Im replacing \Re). Taking $m = \alpha$, we get

$$|G_{1-h}^{(m)}(x_j)| = |\Re f(x_j + iy)|.$$

The expansion at the cusp a_j/b_j implies that, for some $\mu \in \mathbf{C} - \{0\}$,

$$f(x_j + iy) \sim \mu(b_j(x_j + iy) - a_j)^{-r} = \mu(b_j y)^{-r}(\tau_j + i)^{-r}$$

as $y \rightarrow 0^+$, and a suitable choice of τ_j gives

$$|\Re f(x_j + iy)| > C y^{-r/2-\varepsilon r}.$$

Hence we can take $\beta_2 = \alpha - r/2 - \varepsilon r$ in Proposition 3.10, getting

$$\dim(\mathcal{G}) \geq \max(1, 2 + r/2 - \alpha + (r + 2)\varepsilon),$$

and the result is obtained as $\varepsilon \rightarrow 0^+$.

If α is not integral and $[\alpha] + 1$ is even (again, the odd case is completely similar), using the gamma integral we have for $m = [\alpha] + 1$

$$(4.10) \quad |G_{1-h}^{(m)}(x_j)| = |\Re I(x_j)|,$$

where

$$I(x_j) = \frac{(2\pi)^\beta}{\Gamma(\beta)} \int_0^\infty t^{\beta-1} f'(x_j + it + iy) dt \quad \text{with } \beta = \alpha + 1 - m.$$

Let I_1 and I_2 be the contributions to the integral for $0 < t < T$ and $t > T$, respectively, *i.e.*,

$$(4.11) \quad I(x_j) = \frac{(2\pi)^\beta}{\Gamma(\beta)} (I_1 + I_2).$$

Integrating I_2 by parts, we get

$$I_2 = iT^{\beta-1}(f(x_j + iy + iT) - a_0) + i(\beta - 1) \int_T^\infty t^{\beta-2}(f(x_j + it + iy) - a_0) dt.$$

By Lemma 3.5 we have

$$|f(x_j + it + iy) - a_0| < C \min(b_j^{-r}(t + y)^{-r} + b_j^r, (t + y)^{-r}),$$

which when substituted into the preceding formula for I_2 with the choice $T = y^{1-2\varepsilon}$ gives

$$(4.12) \quad |I_2| = O(y^\delta)$$

with $\delta = \beta - 1 - r/2 + \varepsilon(2 + r - 2\beta)$.

To treat I_1 , note that upon differentiating the expansion of f at the cusp a_j/b_j we obtain for $z = x_j + iy + it$, $0 < t < T$,

$$f'(z) = -r\mu b_j(b_j z - a_j)^{-r-1} + O(|b_j z - a_j|^{-r-2} e^{-C(y+t)|b_j z - a_j|^{-2}}).$$

For t far from T (say $t < y^{1-\varepsilon}$) the error term is exponentially small, and a calculation proves that its total contribution to I_1 is absorbed by $O(y^\delta)$. Hence

$$\begin{aligned} I_1 &= -r\mu b_j \int_0^T t^{\beta-1} (b_j \tau_j y + b_j(y+t)i)^{-r-1} dt + O(y^\delta) \\ &= -r\mu b_j^{-r} y^{-r-1+\beta} \int_0^{y^{-2\varepsilon}} u^{\beta-1} (\tau_j + i + iu)^{-r} dt + O(y^\delta). \end{aligned}$$

With a suitable choice of τ_j we have

$$|\Re I_1| > C y^{\beta-1-r/2-r\varepsilon}.$$

Substituting this and (4.12) into (4.11) and (4.10), we conclude that

$$|G_{1-h}^{(m)}(x_j)| > C' y^{\alpha-m-r/2-r\varepsilon}.$$

Hence we can take $\beta_2 = \alpha - r/2 - \varepsilon r$ in Proposition 3.10, as in the case with α integral, and the proof is completed in the same way. \square

§5. THE THETA FUNCTION CASE

In this section we shall apply our results in the context of theta functions, which play an important role in the theory of automorphic forms, providing some simple and explicit examples.

Given a positive definite quadratic form in \mathbf{R}^k with integral coefficients, $Q(\vec{x}) = \vec{x}^t M \vec{x}$, the corresponding theta function,

$$\Theta(z) = \sum_{\vec{n} \in \mathbf{Z}^k} e(Q(\vec{n})z),$$

belongs to $\mathcal{M}_{k/2}(\Gamma_0(2N), m)$, where $N = \det(2M)$ and m is a certain multiplier system (see Th. 10.8 in [10]). The computation of its expansion at the cusp $\mathfrak{a} = a/b$ gives that Θ is cuspidal at \mathfrak{a} if and only if

$$S(a, b) = \sum_{\vec{n} \bmod b} e\left(\frac{aQ(\vec{n})}{b}\right) = 0$$

(see §10.3 in [10], especially 10.25), where $\vec{n} \bmod b$ means that each coordinate of \vec{n} runs over a complete residue class modulo b .

Proposition 5.1. *Let $r_Q(n) = |\{\vec{x} \in \mathbf{Z}^k : Q(\vec{x}) = n\}|$ with Q as before. Let $k/2 < \alpha < k/2 + 1$ if $k > 1$, and $3/4 < \alpha < 3/2$ if $k = 1$. Then the function*

$$F(x) = \sum_{n=1}^{\infty} \frac{r_Q(n)}{n^\alpha} e(nx)$$

is differentiable at $x_0 \in \mathbf{Q}$ if and only if $S(1, b) = 0$, where b is the denominator of x_0 (written in lowest terms).

Using a well known formula for Gauss sums, we immediately deduce

Corollary 5.1.1. *Let $r_k(n)$ be the number of representations of n as a sum of k squares. Then, for α as before, the function*

$$F(x) = \sum_{n=1}^{\infty} \frac{r_k(n)}{n^\alpha} e(nx)$$

is differentiable at $x_0 \in \mathbf{Q}$ if and only if $x_0 = a/(2b)$ with a and b odd integers.

Now we consider the special case of functions defined in Proposition 5.1 for $k = 1$ and $k = 2$. In both cases Theorem 2.3 applies, proving that the fractal dimension of their graphs is $\max(1, 2 - \alpha + k/4)$.

Note that the simplest case, $k = 1$, includes Riemann's example. To keep the analogy with Corollary 2.2.2, we shall express the value of the derivative at $1/2$ in terms of the Riemann ζ -function.

Proposition 5.2. *Consider the function*

$$F(x) = \sum_{n=1}^{\infty} \frac{e(n^2 x)}{n^{\beta}}.$$

Then for $\beta > 3/2$

$$F'(1/2) = 2\pi i(2^{3-\beta} - 1)\zeta(\beta - 2).$$

With a closer analysis of the binary case we shall show that a direct evaluation of $S(1, b)$ can be avoided in Proposition 5.1.

Proposition 5.3. *Consider a binary integral definite positive quadratic form $Q(x, y) = Ax^2 + Bxy + Cy^2$. Let 2^r and 2^s be the highest powers of 2 dividing the discriminant $D = B^2 - 4AC$ and $\gcd(A, B, C)$, respectively; then the function*

$$F(x) = \sum_{n=1}^{\infty} \frac{r_Q(n)}{n^{\alpha}} e(nx), \quad 1 < \alpha < 3/2,$$

(and its real and imaginary parts) is continuous but nowhere differentiable if and only if $r = 2s$. Otherwise, F is differentiable at x_0 if and only if $x_0 = 2^{-s-1}a/b$ or $x_0 = 2^{1+s-r}a/b$ with a and b odd integers.

Examples. When (n, m) runs over $\mathbf{Z}^2 - \{(0, 0)\}$, the function

$$\sum_{n,m} \frac{e((n^2 + mn + m^2)x)}{(n^2 + mn + m^2)^{5/4}}$$

is nowhere differentiable because $r = s = 0$, and

$$\sum_{n,m} \frac{e((n^2 + 2m^2)x)}{(n^2 + 2m^2)^{5/4}}$$

is differentiable at $x_0 = a/b$ if and only if $b \equiv 2, 4, 6 \pmod{8}$ and a is odd, because $r = 3$, $s = 0$.

Given a primitive character ψ with conductor $|\psi|$ and k a positive integer, the associated theta function

$$\Theta(z, k, \psi) = \sum_{n=-\infty}^{\infty} \psi(n) e(kn^2 z)$$

belongs to $\mathcal{M}_{1/2}(\Gamma_0(4N), \chi)$, where $N = k|\psi|^2$ and $\chi = \left(\frac{D}{\cdot}\right)\psi$ with D the discriminant of $\mathbf{Q}(\sqrt{k})$. An important result due to J.-P. Serre and H. Stark [14] asserts that, for any character χ , $\mathcal{M}_{1/2}(\Gamma_0(4N), \chi)$ is generated (as a vector space) by all of the theta functions of this type, and $S_{1/2}(\Gamma_0(4N), \chi)$ is obtained if ψ is restricted to have some odd p -part. (Note that we can assume that ψ is globally even, *i.e.* $\psi(-1) = 1$, because otherwise $\Theta(\cdot, k, \psi)$ vanishes identically.)

Examples. Consider $\psi = \chi_3 \cdot \chi_4$, where χ_3 and χ_4 are the (only) primitive characters of conductor 3 and 4, as they are not even. Then

$$\Theta(z, 1, \psi) = \sum_{n=-\infty}^{\infty} \psi(n) e(n^2 z) \in S_{1/2}(\Gamma_0(576), \psi).$$

By Corollary 2.2.1 and Corollary 2.2.4, the function

$$\sum_{n=-\infty}^{\infty} \psi(n) \frac{e(n^2 x)}{n^\beta} = \sum_{n \equiv \pm 1 \pmod{12}} \frac{e(n^2 x)}{n^\beta} - \sum_{n \equiv \pm 5 \pmod{12}} \frac{e(n^2 x)}{n^\beta}, \quad 3/2 < \beta < 5/2,$$

and its real and imaginary parts are differentiable in \mathbf{Q} and not differentiable in $\mathbf{R} - \mathbf{Q}$.

If $\psi = \left(\frac{\cdot}{5}\right)$, then, as ψ is even and $|\psi| = 5$,

$$\Theta(z, 1, \psi) = \sum_{n=-\infty}^{\infty} \psi(n) e(n^2 z) \in \mathcal{M}_{1/2}(\Gamma_0(100), \psi) - S_{1/2}(\Gamma_0(100), \psi).$$

In fact it can be checked that it is cuspidal exactly at the non-equivalent cusps 0, 1/2, 1/4, 1/10, 7/10, 9/10, 1/25, 1/50 and 1/100. By Theorem 2.2, the function

$$\sum_{n \equiv \pm 1 \pmod{5}} \frac{e(n^2 x)}{n^\beta} - \sum_{n \equiv \pm 2 \pmod{5}} \frac{e(n^2 x)}{n^\beta}, \quad 3/2 < \beta < 5/2,$$

and its real and imaginary parts are differentiable at x_0 if and only if x_0 belongs to the $\Gamma_0(100)$ -orbit of these cusps, which can be expressed in terms of congruence conditions as in Riemann's example.

We finish by giving the proofs of the results of this section.

Proof of Proposition 5.1. Note first that, by considering the Galois automorphism that sends $e(1/b)$ into $e(a/b)$, for $\gcd(a, b) = 1$ we have

$$(5.1) \quad S(1, b) = 0 \Leftrightarrow S(a, b) = 0.$$

Hence $S(1, b) \neq 0$ implies $S(a, b) \neq 0$, and Θ is not cuspidal at $x_0 = a/b$. For $k = 1$, partial summation (use the periodicity of $S(\cdot, b)$) proves that the series formally defining $F'(x_0)$ is not even Abel summable for $\alpha < 3/2$; hence, by III.7.6 in [17], F is not differentiable at x_0 (cf. [3]). This is also the case for $k > 1$ when $\alpha < k/2 + 1$, because the expansion at the cusp x_0 implies that the gamma integral representation

$$F'(x_0 + it) = \frac{(2\pi)^\alpha i}{\Gamma(\alpha - 1)} \int_0^\infty t^{\alpha-2} (\Theta(x_0 + i(t+y)) - 1) dt$$

diverges as $y \rightarrow 0^+$ (compare with the proof of Proposition 3.1).

Finally, if $S(1, b) = 0$, then Θ is cuspidal at $x_0 = a/b$ and, by the formula following Theorem 2.2, F is differentiable at x_0 in the indicated range. \square

Proof of Proposition 5.2. An elementary computation proves that for $\Re s > 1/2$

$$-\frac{1}{1^{2s}} + \frac{1}{2^{2s}} - \frac{1}{3^{2s}} + \frac{1}{4^{2s}} - \dots = (2^{1-2s} - 1)\zeta(2s),$$

and by the gamma integral

$$\frac{(-1)^n}{n^{\beta-2}} = \frac{(2\pi)^{\beta/2-1}}{i\Gamma(\beta/2)} \int_0^\infty t^{\beta/2-1} 2\pi i n^2 e(n^2(1/2 + it)) dt.$$

Hence

$$\frac{(2\pi)^{\beta/2}}{\Gamma(\beta/2)} \int_0^\infty t^{\beta/2-1} \Theta'(1/2 + it) dt = 2\pi i (2^{3-\beta} - 1) \zeta(\beta - 2)$$

and, according to Theorem 2.2, the left hand side is $F'(1/2)$. \square

Proof of Proposition 5.3. With a change of variable $x \mapsto 2^{-s}x$ we can always assume $s = 0$.

Writing $\vec{n} = b_1\vec{n}_1 + b_2\vec{n}_2$ with $\vec{n}_1 \pmod{b_2}$ and $\vec{n}_2 \pmod{b_1}$, it is easy to check that

$$S(1, b_1b_2) = S(b_2, b_1)S(b_1, b_2)$$

for $\gcd(b_1, b_2) = 1$.

As Q is diagonalizable in $\mathbf{Z}/p^m\mathbf{Z}$ for p an odd prime (by completing squares), the explicit evaluation of Gauss sums can be applied to get

$$S(1, p^m) \neq 0.$$

Hence, by (5.1), the above multiplicative property, Theorem 2.2 and Corollary 2.2.4, $\Re F$ and $\Im F$ are differentiable at x_0 if and only if for the highest power of 2 dividing the denominator of x_0 , say 2^t , we have

$$S(1, 2^t) = 0.$$

Therefore it is enough to prove that this relation does not hold if $r = 0$ and, for $r \neq 0$, it is fulfilled if and only if $t = 1$ or $t = r - 1$.

If $r \neq 0$, then B is even, $4|D$ and (perhaps exchanging A and C) we can assume that A is odd. Hence Q can be diagonalized in $\mathbf{Z}/2^t\mathbf{Z}$ (again by completing squares) as $Ax^2 + A'(D/4)y^2$ with A and A' odd. So, by the multiplicative property and (5.1),

$$S(1, 2^t) = 0 \Leftrightarrow \sum_{n \pmod{2^t}} e\left(\frac{n^2}{2^t}\right) \cdot \sum_{m \pmod{2^{t+2}}} e\left(\frac{Dm^2}{2^{t+2}}\right) = 0,$$

which, by the explicit evaluation of Gauss sums, holds only for $t = 1$ or $t = r - 1$ (note that $r > 1$).

It remains only to prove that $S(1, 2^t) \neq 0$ if $r = 0$. We proceed by induction on t . For $t = 0$ or $t = 1$, the result is trivial (note that B is odd). If $t > 1$, writing $\vec{n} = 2^{t-1}\vec{a} + \vec{b}$ with $\vec{a} = (a_1, a_2)$, $\vec{b} = (b_1, b_2)$, we have

$$S(1, 2^t) = \sum_{\vec{b} \pmod{2^{t-1}}} \sum_{\vec{a} \pmod{2}} e\left(\frac{B(a_1b_2 + a_2b_1)}{2}\right) e\left(\frac{Q(\vec{b})}{2^t}\right).$$

As the innermost sum vanishes if b_1 and b_2 are not simultaneously even, we conclude that $S(1, 2^t) = 2S(1, 2^{t-2})$. \square

§6. SOME GRAPHICS

In this section we present several pictures of some of the foregoing functions. Regarding the computational aspects, the Fourier coefficients of the automorphic forms corresponding to elliptic curves were obtained by use of the PARI package (the author thanks E. González Jiménez for his invaluable help), and partial sums have been calculated with simple FORTRAN programs, giving data output that was finally plotted using the MATLAB package.

First of all, we consider one of the “simplest” elliptic curves: one with minimal conductor $N = 11$, given by the equation

$$y^2 + y = x^3 - x^2 - 10x - 20.$$

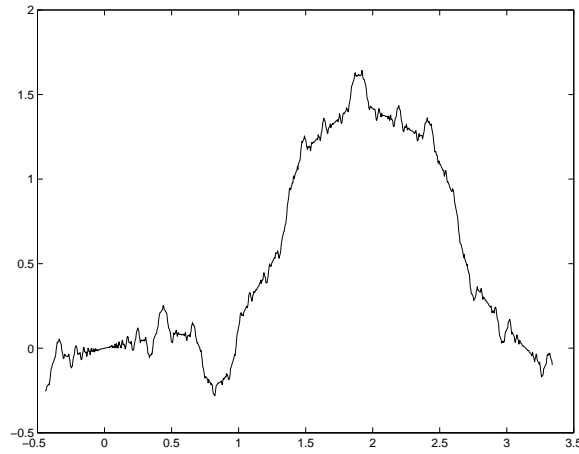


FIGURE 1.

Let a_n be the coefficients of its Hasse-Weil L -function, and let

$$g(x) = \sum_{n=1}^{\infty} \frac{a_n}{n^{7/4}} \sin(nx).$$

According to Proposition 2.4, g is differentiable exactly at rational multiples of π , and its graph has dimension $5/4$. Plotting g , we obtain Figure 1.

We can see the differential behavior at $x = 0$ as a quick oscillation with shrinking amplitude around the origin. The same phenomenon can be observed or guessed in this picture, in a self-similar way, around $2\pi/11$, $4\pi/11$, $2\pi/4$, $6\pi/11$, $2\pi/3$, $8\pi/11$, $10\pi/11$ and π . With the help of a computer we can check that by enlarging the drawing (typically with a very big scale) the same kind of oscillatory behavior appears around these and other rational multiples of π . For instance, the enlarged graphs around $x = \pi$ and $x = 8\pi/11$ are shown in Figure 2 (note the different scale).

A natural question is, Why can we observe the oscillation showing differentiability to the naked eye only at the aforementioned points on the original whole graph of g ? An answer can be provided by examining the use of Lemma 3.3 in the proof of Theorem 2.2: note that if z tends (non-tangentially) to a cusp a/b then, according to Lemma 3.3, a cusp form $f(z)$ goes to zero more quickly when the denominator b is smaller and when the width of the cusp, \sqrt{K} , is larger. In our case, the underlying group is $\Gamma = \Gamma_0(11)$, and there are only two cusps up to equivalence by Γ , one of width 1 and one of width 11. The cusps $x = 0, 1/4, 1/3, 1/2$ are equivalent to the first one, and $x = 1/11, 2/11, 3/11, 4/11, 5/11$ are equivalent to the second. Hence the magnifying factor needed to observe the graph of $g(2\pi x)$ around these latter points is, roughly speaking, one eleventh the size which should correspond to the size of their denominators.

Under the (weak) Birch-Swinnerton-Dyer conjecture (see [9]), using Corollary 2.2.2, we have that an elliptic curve over \mathbf{Q} contains infinitely many rational points if and only if B_2 (defined as in Proposition 2.4) is flat around the origin, *i.e.*, $B'_2(0) = 0$. This allows a geometrical representation of the conjecture. For instance,

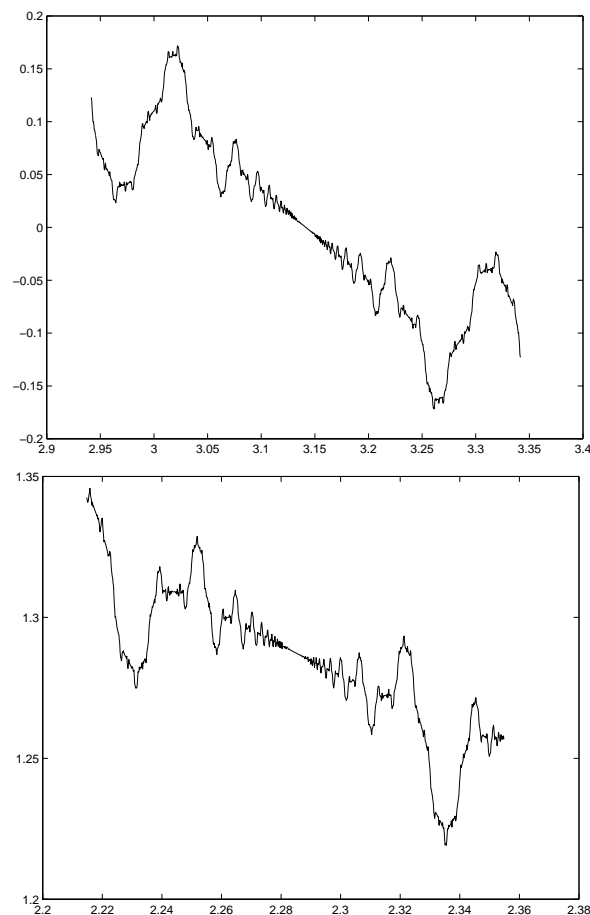


FIGURE 2.

consider the elliptic curves

$$E_1 : y^2 + y = x^3 + 4x - 6, \quad E_2 : y^2 + y = x^3 - x.$$

The graphs of B_2 in a neighborhood of the origin are shown respectively in Figure 3, which reflects the fact that E_1 contains only finitely many rational points while E_2 contains infinitely many. The scale factor to see this phenomenon depends on the size of the conductor.

The theta function case produces examples that are easily generated with a computer. One of the simplest is Riemann's example, shown in Figure 4. The graph illustrates clearly the differentiability of the function at $x = 1/2$ and its absence at $x = 0$.

One can study the nature of the oscillatory behavior around points of differentiability. Heuristically, if the cusp $\mathfrak{a} = 0$ is a differentiability point of f_α , the expansion of f at this cusp suggests an oscillation with varying frequencies of the type $e(C/x)$ (note that $\sigma_{\mathfrak{a}}z = -C/z$), and the formula after Theorem 2.2 suggests that, in suitable ranges, the amplitude has the order $|x|^{2\alpha-r}$. Hence an approximation like $f_\alpha(0) + f'_\alpha(0)x + C'|x|^{2\alpha-r}e(C/x)$ could be expected for x small enough,

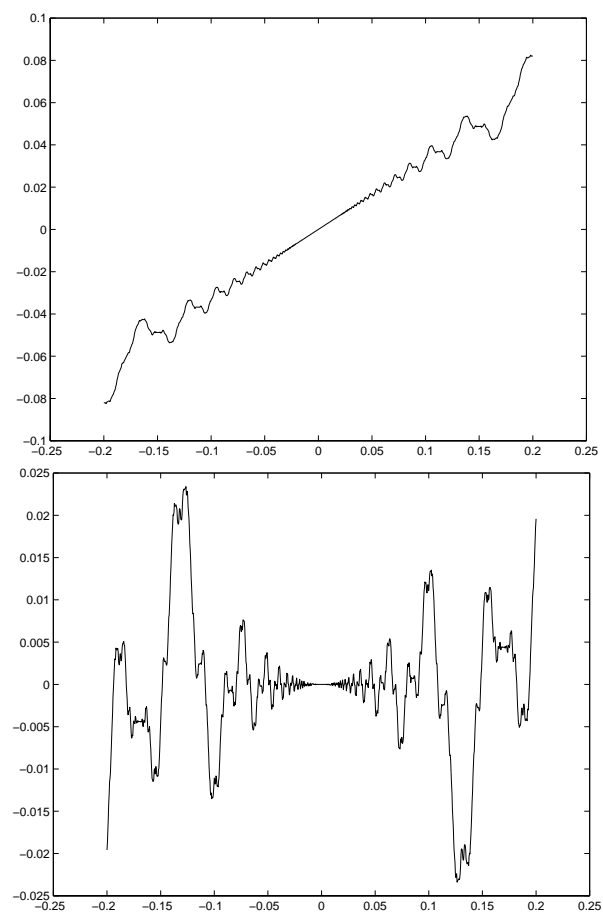


FIGURE 3.

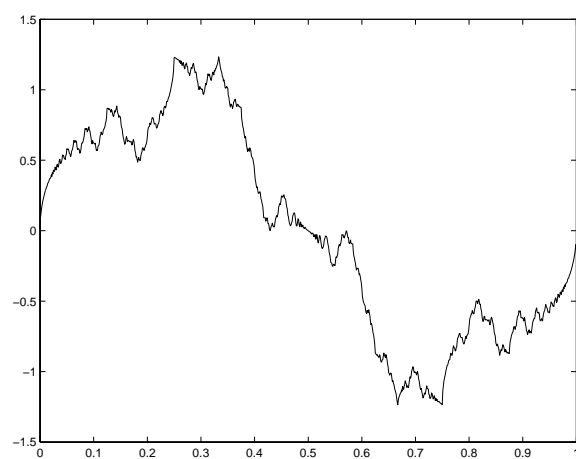


FIGURE 4.

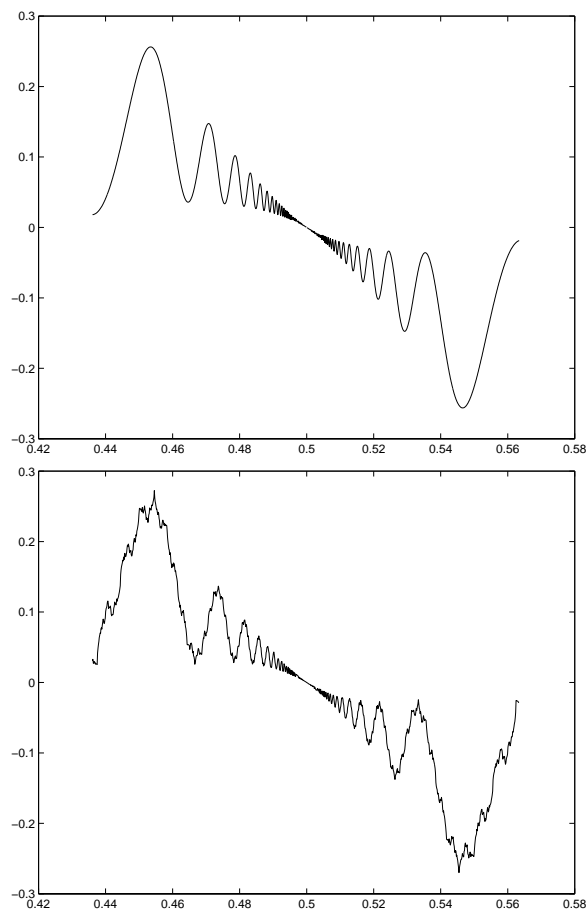


FIGURE 5.

and similar expansions on small neighborhoods of other differentiability points. For instance, for Riemann's example $\alpha = 1$, $r = 1/2$, $f_\alpha(1/2) = 0$ and $f'_\alpha(1/2) = -\pi$. Figure 5 shows the graph of $g(x) = -\pi(x - 1/2) + 4|2x - 1|^{3/2} \sin(2x - 1)^{-1}$ and the actual graph of the Riemann's example around $x = 1/2$. It is clear that they have the same behavior.

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